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Delay-Feedback Using Derivatives for Minimal Time Linear Control Problems

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A modification of Popov's linear minimal time problem is solved by allowing delay-feedback in both the state and its derivative. It is shown how to control the state of a linear time invariant differential equation to any nontrivial subspace of the state space and remain there for all future time. A technique is given for constructing the feedback law which allows the system's trajectory to reach the given subspace in minimum time.

In [4] Popov introduced the following variant of the classical minimal time problem: For the linear time invariant differential equation $\dot{x} = Ax + u(t)$ find a control law of the form $u(t) = Bx(t - h)$ such that all solutions of the closed loop system $\dot{x}(t) = Ax(t) + Bx(t - h)$ reach the subspace $q^*x(t_1) = 0$ in minimal time t_1 and remain on this subspace for all future time. Popov gave an elegant solution to this problem, showed how it is possible to construct the B matrix, and proved that the minimal time is $t_1 = 2h$.

In the present paper we effectively solve an obvious modification of Popov's problem by allowing delay-feedback in both the state and its derivative. With the introduction of several lag terms we show how it is possible to reach any nontrivial subspace of the state space in minimal time h and remain on the given subspace for any time greater than h. The variant of the minimal time problem that we consider follows.

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PROBLEM. Given a triple (A, Q, h) where A is a complex matrix of order n, Q an $n \times p$ nonzero matrix, and h a positive scalar. Find $n \times nm$ matrices

$$B = (B_1 B_2 \cdots B_m), \qquad C = (C_1 C_2 \cdots C_m)$$

such that

(i) all solutions of the neutral type delay-differential system [2]

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{m} [B_j x(t - jh) + C_j \dot{x}(t - jh)], \qquad (t > 0) \tag{1}$$

satisfy the degeneracy property

$$Q^* x(t) \equiv 0 \qquad \text{for } t \geqslant t_1 \tag{2}$$

and

(ii) t_1 is minimal. That is, from among the class of couples (B, C) that satisfy property (i) find a couple $(\tilde{B}, \tilde{C})$ that minimizes t_1.

Remark. If the degeneracy property (2) is satisfied then any linear combination of the rows of Q^* will also satisfy (2). Thus, without any loss of generality, we assume $\operatorname{rank}(Q^*) = p$. The degeneracy property occurring in (i) has been investigated for the situation $m = 1$ by Choudhury [3].

For convenience, define $C_0 = -I$, where I is the identity matrix, and $C_{m+1} = 0$. Then a solution to the stated problem is contained in the following.

THEOREM 1. *Equation* (1) *satisfies the degeneracy property* $Q^* x(t) \equiv 0$ *for* $t \geqslant h$, *if and only if,*

$$\begin{aligned} Q^* e^{Ah} C_{j-1} &= Q^* C_j \qquad (j = 1, \dots, m+1), \\ C_0 &= -I, \qquad C_{m+1} = 0, \end{aligned} \tag{3}$$

and

$$B_j + AC_j = D_j \qquad (j = 1, \dots, m), \tag{4}$$

where D_j *is any matrix that satisfies*

$$Q^* e^{At} D_j \equiv 0. \tag{5}$$

Further, the minimal time is h.

Proof. For any solution x of (1) define

$$y(t) = x(t) - \sum_{j=1}^{m} C_j x(t - jh).$$

Then Eq. (1) can be expressed as

$$\dot{y}(t) = Ay(t) + \sum_{j=1}^{m} (B_j + AC_j)\, x(t - jh).$$

The solution of this equation is

$$y(t) = e^{At}y(0) + \sum_{j=1}^{m} \int_0^t e^{A(t-s)}(B_j + AC_j)\, x(s - jh)\, ds$$

and for $t \geqslant h$

$$y(t - h) = e^{A(t-h)}y(0) + \sum_{j=1}^{m} \int_0^{t-h} e^{A(t-h-s)}(B_j + AC_j)\, x(s - jh)\, ds.$$

Multiply this last equation by e^{Ah} and subtract the result from the previous equation to obtain

$$y(t) - e^{Ah}y(t - h) = \sum_{j=1}^{m} \int_{t-h}^{t} e^{A(t-s)}(B_j + AC_j)\, x(s - jh)\, ds \qquad (t \geqslant h).$$

If we multiply this equation by Q^* and use the definition of y we obtain for $t \geqslant h$

$$\begin{aligned} Q^*x(t) - \sum_{j=1}^{m+1} Q^*(C_j - e^{Ah}C_{j-1})x(t - jh) \\ = \sum_{j=1}^{m} \int_{t-h}^{t} Q^*e^{A(t-s)}(B_j + AC_j)\, x(s - jh)\, ds. \end{aligned} \tag{6}$$

Now use (3)–(5) in (6) to obtain $Q^*x(t) \equiv 0$ for $t \geqslant h$.

To show necessity we assume $Q^*x(t) \equiv 0$ for $t \geqslant h$. In particular at $t = h$, Eq. (6) becomes

$$\begin{aligned} \sum_{j=1}^{m} \int_0^h Q^*e^{A(h-s)}(B_j + AC_j)x(s - jh)\, ds \\ = - \sum_{j=1}^{m+1} Q^*(C_j - e^{Ah}C_{j-1})\, x(h - jh). \end{aligned}$$

Since this equation is to be satisfied for all initial functions x, we deduce by selecting suitable initial functions (e.g. test functions [5]) that

$$Q^*e^{A(h-s)}(B_j + AC_j) \equiv 0$$

and

$$Q^*(C_j - e^{Ah}C_{j-1}) = 0 \qquad (j = 1,..., m + 1).$$

To show that h is minimal, we take $t_1 \in [0, h)$ and consider any initial function that is zero for $t \in [-mh, t_1 - h]$ and takes on the value $e^{A^* t_1} q$ at $t = 0$. In the interval $(0, t_1]$, Eq. (1) reduces to $\dot{x} = Ax$ with solution $x(t) = e^{At}x(0)$. If q is any column of Q we have

$$q^* x(t_1) = \| q^* e^{At_1} \|^2 \neq 0$$

which shows that $t_1 = h$ is minimal.

COROLLARY 1. *If* (Q^*, A) *is completely observable (c.o.) then* $B_j = -AC_j$. *The proof follows directly from the definition of c.o.* (see e.g. [5]) *and from Eq.* (5).

COROLLARY 2. $\operatorname{rank}(Q) < n$. *Otherwise, using* (3), $C_j = -e^{jAh}$ *and* $C_{m+1} \neq 0$.

If the triple (A, Q, h) is known a priori, then the matrices B and C can be constructed provided (3)–(5) have a solution. It is clear that once C is constructed from (3), D_j from (5), then B follows from (4). It is possible to give a n.a.s.c. for the existence of a C matrix which satisfies (3) in the terminology of linear system theory. For this purpose, let N denote an $n \times (n - p)$ matrix whose columns are a basis for the null space of Q^*, i.e., the subspace we wish to reach in minimal time.

THEOREM 2. *Equation* (3) *has a solution for some* m, *if and only if, the couple* (e^{Ah}, N) *is completely controllable.*

Proof. Equation (3) is the equivalent of saying that every column of the matrix $C_j - e^{Ah}C_{j-1}$ ($j = 1,\ldots, m + 1$) lies in the null space of Q^*. Thus the most general $n \times n$ matrices which satisfy (3) are of the form NF_j ($j = 1,\ldots, m + 1$) where F_j are $(n - p) \times n$ matrices. If we eliminate C_j from the equations

$$C_j - e^{Ah}C_{j-1} = NF_j \qquad (j = 1,\ldots, m + 1) \tag{7}$$

we obtain

$$NF_{m+1} + e^{Ah}NF_m + \cdots + e^{mAh}NF_1 = e^{(m+1)Ah}. \tag{8}$$

The existence of solutions of (8) is necessary and sufficient for the solvability of (3). The matrix $e^{(m+1)Ah}$ is nonsingular, hence there exist matrices F_j which satisfy (8), if and only if,

$$\operatorname{rank}(N e^{Ah}N \cdots e^{mAh}N) = n. \tag{9}$$

Therefore, using the Cayley–Hamilton theorem, (3) has a solution, if and only if,

$$\operatorname{rank}(Ne^{Ah}N \cdots e^{(n-1)Ah}N) = n.$$

From Eq. (9) we immediately have the following.

COROLLARY 3. *If* (3) *has a solution, then the minimal number of lag terms required is*

$$m = \min_{m' \in I}\{m' \mid \operatorname{rank}(Ne^{Ah}N \cdots e^{m'Ah}N) = n\}$$

and

$$m \geqslant p(n-p)^{-1}.$$

Remark. If $A = \alpha I$, α a scalar, then

$$\operatorname{rank}(Ne^{\alpha h}N \cdots e^{m\alpha h}N) = n - p < n$$

and degeneracy at h is impossible. The reader may wish to compare this with the system ($A = 0$)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1(t-2) \\ x_2(t-2) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-3) \\ x_2(t-3) \end{bmatrix}$$

which is degenerate with respect to $Q^* = (1\ 1)$ for $t \geqslant 3$ [1].

EXAMPLES

EXAMPLE 1.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad Q^* = (1 \quad 1), \quad h = 1.$$

We calculate

$$N^* = (1, -1) \qquad \text{and} \qquad e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}.$$

Since $\operatorname{rank}(Ne^{A}N) = 2$ the minimal m required is $m = 1$. For this example, we prefer to show dependence on parameters by arbitrarily letting $m = 3$. Equation (8) becomes

$$\begin{bmatrix} 1 & 0 & -1 & -2 \\ -1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} F_4 \\ F_3 \\ F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

with the solution

$$\begin{aligned} F_4 &= (\ \ 1 + \ \alpha_1 + 2\alpha_2\,, \quad\ \ 4 + \ \alpha_3 + 2\alpha_4) \\ F_3 &= (-1 - 2\alpha_1 - 3\alpha_2\,, \quad -5 - 2\alpha_3 - 3\alpha_4) \\ F_2 &= (\qquad \alpha_1 \qquad , \qquad\quad \alpha_3 \qquad) \\ F_1 &= (\qquad \alpha_2 \qquad , \qquad\quad \alpha_4 \qquad). \end{aligned}$$

Using (7):

$$C = (C_1C_2C_3) = \begin{bmatrix} -1+\alpha_2 & -1+\alpha_4 & -1+\alpha_1 & -2+\alpha_3 & -2-2\alpha_1-4\alpha_2 & -8-2\alpha_3-4\alpha_4 \\ -\alpha_2 & -1-\alpha_4 & -\alpha_1-\alpha_2 & -1-\alpha_3-\alpha_4 & 1+\alpha_1+2\alpha_2 & 4+\alpha_3+2\alpha_4 \end{bmatrix}.$$

$D_j = 0$ because (Q^*, A) is c.o., hence $B = -AC$.

Example 2. $A = (0, e_1, e_2, e_3)$, $Q^* = \left[\begin{smallmatrix} e_2^* \\ e_3^* \end{smallmatrix}\right]$ where e_i are unit vectors in R^4, and $h = 1$. Then $N = (e_1, e_4)$,

$$\operatorname{rank}(N, e^A N) = 3 \qquad \text{and} \qquad \operatorname{rank}(N, e^A N, e^{2A} N) = 4.$$

Thus the minimal m is equal to two and $m > p(n-p)^{-1}$ (see Corollary 3). We also note that $Q^*A^3 = 0$, therefore the pair (Q^*, A) is not observable. Using (7),

$$C = (C_1C_2) = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\ 0 & -1 & -1 & -\frac{1}{2} & 0 & -\frac{1}{2} & -\frac{3}{4} & -\frac{1}{2} \\ 0 & 0 & -1 & -1 & 0 & 1 & \frac{3}{2} & 1 \\ 0 & 1 & \frac{5}{2} & 2 & 0 & -1 & -\frac{3}{2} & -1 \end{bmatrix}$$

From (5), $D_j = e_1 d^*$ where d is an arbitrary vector. We calculate from (4)

$$B = (B_1B_2)$$

$$= \begin{bmatrix} \alpha_9 & 1+\alpha_{10} & 1+\alpha_{11} & \frac{1}{2}+\alpha_{12} & \alpha_{13} & \frac{1}{2}+\alpha_{14} & \frac{3}{4}+\alpha_{15} & \frac{1}{2}+\alpha_{16} \\ 0 & 0 & 1 & 1 & 0 & -1 & -\frac{3}{2} & -1 \\ 0 & -1 & -\frac{5}{2} & -2 & 0 & 1 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is clear that the "solution" to the minimal time problem is, in general, not unique. This suggests that we may require, for example, the solutions of Eq. (1) to satisfy an additional criterion.

Conclusions

A technique is given for construction of a feedback law for a linear time invariant linear system using lag terms. The closed loop system will reach (and remain on) any prescribed nontrivial subspace in minimal time. A necessary and sufficient condition for this degeneracy property is given.

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